Electromagnetic field tensor.

We have the transformation relation relating the primed and unprimed systems through the em fields are as follows.

$$\left.\begin{aligned} \bar{E}_x &= E_x \\ \bar{E}_y &= \gamma(E_y - vB_z) \\ \bar{E}_z &= \gamma(E_z + vB_y) \end{aligned}\right\} \rightarrow (1A) \qquad \left.\begin{aligned} \bar{B}_x &= B_x \\ \bar{B}_y &= \gamma\left(B_y + \frac{v}{c}E_z\right) \\ \bar{B}_z &= \gamma\left(B_z - \frac{v}{c}E_y\right) \end{aligned}\right\} \rightarrow (1B) \rightarrow (1)$$

If we look through the transformation relations representing by equation (1), we see the E and B components are entangled together when we go from one inertial frame to another inertial frame. If the tranformation relation of E and B is represented by a single equation just like the spatial and temporal part of 4 vectors, in tensor form the tranformation matrix will be a antisymmetric second rank tensor.

A second rank tensor is an object with two indices which transform with two factors of Λ as

$$\bar{T}^{\mu\nu} = \Lambda^{\mu}_{\alpha}\Lambda^{\nu}_{\beta}T^{\alpha\beta} \rightarrow (2)$$

using Einstein summation convension.

In 4 d system A tensor has $4 \times 4 = 16$ components and can be represented as

$$T^{\mu\nu} = \begin{Bmatrix} t^{00} & t^{01} & t^{02} & t^{03} \\ t^{10} & t^{11} & t^{12} & t^{13} \\ t^{20} & t^{21} & t^{22} & t^{23} \\ t^{30} & t^{31} & t^{32} & t^{33} \end{Bmatrix} \rightarrow (3)$$

The symmetric tensor has the property $t^{\mu\nu} = t^{\nu\mu}$ and the antisymmetric tensor has the property

$$t^{\mu\nu} = -t^{\nu\mu}.$$

Thus the tensor (3) to be antisymmetric the form of the tensor should be

$$t^{\mu\nu} = \begin{Bmatrix} 0 & t^{01} & t^{02} & t^{03} \\ -t^{01} & 0 & t^{12} & t^{13} \\ -t^{02} & -t^{12} & 0 & t^{23} \\ -t^{03} & -t^{13} & -t^{23} & 0 \end{Bmatrix} \rightarrow (4)$$

As $t^{\mu\nu} = -t^{\nu\mu}$

Now according to the transformation rule

$$\bar{t}^{01} = \Lambda^0_\alpha \Lambda^1_\beta t^{\alpha\beta}$$

Now $\Lambda^0_\alpha = 0$ unless $\alpha = 0$ or 1

My $\Lambda^1_\beta = 0$ unless $\beta = 0$ or 1.

$$\bar{t}^{01} = \Lambda^0_0 \Lambda^1_0 t^{00} + \Lambda^0_0 \Lambda^1_1 t^{01} + \Lambda^0_1 \Lambda^1_0 t^{10} + \Lambda^0_1 \Lambda^1_1 t^{11}$$

From the general form of an anticymmetric tensor $\Lambda^0_0 = \Lambda^1_1 = 0$ & $\Lambda^1_0 = \Lambda^0_1$

$$t^{00} = t^{11} = 0 \quad \& \quad t^{10} = -t^{01}$$

$$\bar{t}^{01} = \Lambda^0_0 \Lambda^1_1 t^{01} + \Lambda^0_1 \Lambda^1_0 t^{10} = \Lambda^0_0 \Lambda^1_1 t^{01} - \Lambda^1_1 \Lambda^1_0 t^{01}$$

Comparing with the transformation matrix relating to 4D transformation matrix

$$\begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\bar{t}^{01} = (\gamma^2 - (\gamma\beta)^2) t^{01}$$

$$\Rightarrow \bar{t}^{01} = t^{01}$$

In the same way the complete set of transformation for the other components are

$$\bar{t}^{01} = t^{10}$$

$$\bar{t}^{02} = \gamma(t^{02} - \beta t^{12})$$

$$\bar{t}^{03} = \gamma(t^{03} + \beta t^{31})$$

$$\bar{t}^{23} = t^{23}$$

$$\bar{t}^{31} = \gamma(t^{31} + \beta t^{03})$$

$$\bar{t}^{12} = \gamma(t^{12} - \beta t^{02})$$

From the transformation relation of E and B, representing the electric and magnetic components as the components of unique field F as

$$F^{01} = \frac{E_x}{c}, \quad F^{02} = \frac{E_y}{c} \quad \& \quad F^{03} = \frac{E_z}{c}$$

$$F^{12} = B_z \quad F^{31} = B_y \quad \& \quad F^{23} = B_x$$

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The combined transformation relation for the em field in tensor form can be written as

$$F^{\mu\nu} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_y & 0 \end{bmatrix} \rightarrow (5)$$

The electromagnetic field tensor may be constructed taking the equation (1B) which comes as follow

$$G^{\mu\nu} = \begin{bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & +E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{bmatrix} \rightarrow (6)$$

The transition from $F^{\mu\nu}$ to $G^{\mu\nu}$ can be achieved by replacing $\frac{E}{c} = B$, & $B = -E/c$

Both the tensors $F^{\mu\nu}$ and $G^{\mu\nu}$ give the correct transformation for E and B.

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